Linear independence of indefinite iterated Eisenstein integrals

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Abstract

We prove linear independence of indefinite iterated Eisenstein integrals over the fraction field of the ring of formal power series $\mathbb{Z}[[q]]$. Our proof relies on a general criterium for linear independence of iterated integrals, which has been established by Deneufchâtel, Duchamp, Minh and Solomon. As a corollary, we obtain $\mathbb{C}$-linear independence of indefinite iterated Eisenstein integrals, which has applications to the study of elliptic multiple zeta values, as defined by Enriquez.

1 Introduction

Given a collection $\omega_1, \dots, \omega_r$ of smooth one-forms on a smooth manifold M, and a smooth path $\gamma : [0,1] \to M$, one defines their *iterated integral* as

$$\int_\gamma \omega_1 \dots \omega_r = \int_{0 \leq t_1 \leq \dots \leq t_r \leq 1} \gamma^*(\omega_1) \dots \gamma^*(\omega_r), \tag{1.1}$$

where $\gamma^*(\omega_i) = f_i(t_i)\mathrm{d}t_i$ denotes the pull back of ω_i along γ. In the case of a single differential one-form ω, (1.1) is simply the path integral of ω along γ.

A classical application of iterated integrals is the construction of solutions to certain systems of linear differential equations via the method of Picard iteration (cf. e.g. [14]). However, iterated integrals also appear in number theory, prominent examples being *multiple polylogarithms* and *multiple zeta values*, which are iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (see for example the lecture notes [8] for an introduction from the point of view of iterated integrals). It is known that the multiple polylogarithms are linearly independent over $\mathbb{C}$ [13]. Using rather different techniques, this result has been generalized [5], with $\mathbb{C}$ replaced by an arbitrary field of functions satisfying some extra conditions.

On the other hand, another family of iterated integrals arising in number theory are iterated integrals of modular forms. Their study has been initiated by Manin [11], and was later extended in [3, 7, 9]. Known in the literature under the names *iterated Eichler integrals* [3] or *iterated Shimura integrals* [11], these are iterated integrals on the upper half-plane, which generalize the classical Eichler integrals [10], and are also closely related to L-functions of modular forms [11, 3].

Iterated integrals of modular forms also appear in the study of *elliptic multiple zeta values* [4, 6, 2, 12], the latter being a natural genus one analogue of the classical multiple zeta values. In [2], a procedure for decomposing elliptic multiple zeta values into certain $\mathbb{C}$-linear combinations of (indefinite) iterated integrals of Eisenstein series (called iterated Eisenstein integrals for short)[1] is described. The uniqueness of this decomposition, important both for the mathematical theory as well as for applications to physics [1], depended on the $\mathbb{C}$-linear independence of the iterated Eisenstein integrals in question.

[1] All modular forms appearing in this paper are modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$.

In this paper, we prove linear independence of iterated Eisenstein integrals, first over the fraction field $\mathrm{Frac}(\mathbb{Z}[[q]])$ of the ring of formal power series in one variable with integer coefficients, where q is viewed as a coordinate on the open unit disk. By the main result of [5], it is enough to prove that $\mathrm{Frac}(\mathbb{Z}[[q]])$ does not contain primitives of Eisenstein series, which in turn follows from a computation of their denominators.

Having established linear independence over $\mathrm{Frac}(\mathbb{Z}[[q]])$, the linear independence of iterated Eisenstein integrals over $\mathbb{Q}$ follows immediately, since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[[q]])$. Finally, by extending scalars from $\mathbb{Q}$ to $\mathbb{C}$, we obtain the desired $\mathbb{C}$-linear independence of iterated Eisenstein integrals.

Acknowledgments. Very many thanks to Pierre Lochak for bringing the paper [5] to my attention, as well as for helpful discussions and remarks. This paper is part of the author's doctoral thesis at Universität Hamburg, and I would like to thank my advisor Ulf Kühn for helpful remarks.

2 Iterated Eisenstein integrals

2.1. **Eisenstein series**. For $k \geq 1$ denote by G_{2k} the *Hecke-normalized Eisenstein series* (cf. e.g. [17]), which is the function on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, defined by the convergent series

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n \in \mathbb{Q} \oplus q\mathbb{Z}[[q]], \quad q = e^{2\pi i \tau}, \tag{2.1}$$

where B_{2k} denotes the $2k$-th Bernoulli number, and $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$. We also set $G_0 \equiv -1$.

The function G_{2k} is holomorphic, and, for $k \geq 2$, it is a modular form for $\mathrm{SL}_2(\mathbb{Z})$. Write G_{2k}^{∞} for the constant term in its q-expansion, and likewise $G_{2k}^0(q)$ for $G_{2k}(q) - G_{2k}^{\infty}$. Note that for $k \geq 1$, we have

$$G_{2k}^{\infty} = -\frac{B_{2k}}{4k}, \quad G_{2k}^0(q) = \sum_{n \geq 1} \sigma_{2k-1}(n) q^n. \tag{2.2}$$

2.2. **Regularization of iterated integrals**. We would now like to define iterated Eisenstein integrals

$$\int_{\tau}^{i\infty} G_{2k_1}(q_1)\mathrm{d}\tau_1 \dots G_{2k_n}(q_n)\mathrm{d}\tau_n \tag{2.3}$$

as functions depending on some start point $\tau \in \mathbb{H}$, where the integration is performed along some path from τ to the cusp $i\infty$[2]. Unfortunately, in this case the usual definition of iterated integrals (1.1) produces divergent integrals, already in the case of single Eisenstein integrals, i.e. for $n = 1$. In order to overcome this problem, we describe a regularization scheme for such iterated integrals, introduced by Brown in [3]. For the rest of this subsection, we follow [3].

Let $W = \mathbb{C}[[q]]^{<1}$ be the $\mathbb{C}$-algebra of formal power series, which converge on the open q-disk $D = \{q \in \mathbb{C} \mid |q| < 1\}$, and denote by $D^* := D \setminus \{0\}$ the punctured disk. Via the universal covering map

$$\exp : \mathbb{H} \to D^*, \quad \tau \mapsto e^{2\pi i \tau}, \tag{2.4}$$

we can consider W as a $\mathbb{C}$-subalgebra of the $\mathbb{C}$-algebra $\mathcal{O}(\mathbb{H})$ of holomorphic functions on the upper half-plane.

[2]The value of the iterated integral does not depend on the choice of path, since the Eisenstein series are holomorphic functions on a one-dimensional complex manifold.

Write $W = W^0 \oplus W^\infty$ with $W^0 = q\mathbb{C}[[q]]$ and $W^\infty = \mathbb{C}$. For a power series $f \in W$, define f^0 to be its image in W^0 under the natural projection, and define $f^\infty \in W^\infty$ likewise. Denote by $T^c(W)$ the tensor coalgebra on the $\mathbb{C}$-vector space W, which comes equipped with a shuffle product $\sqcup\!\sqcup$. We will use bar notation for elements of $T^c(W)$, and define a map $R : T^c(W) \to T^c(W)$ by the formula

$$R[f_1|\dots|f_n] = \sum_{i=0}^{n} (-1)^{n-i}[f_1|\dots|f_i] \sqcup\!\sqcup [f_n^\infty|\dots|f_{i+1}^\infty]. \tag{2.5}$$

We can now make the

Definition 2.1. Given $f_1, \dots, f_n \in W \subset \mathcal{O}(\mathbb{H})$ as above, their *regularized iterated integral* is defined as

$$I(f_1, \dots, f_n; \tau) := \sum_{i=0}^{n} \int_\tau^{i\infty} R[f_1|\dots|f_i]_{\mathrm{d}\tau} \int_\tau^0 [f_{i+1}^\infty|\dots|f_n^\infty]_{\mathrm{d}\tau}, \tag{2.6}$$

where

$$\int_a^b [f_1|\dots|f_n]_{\mathrm{d}\tau} := \int_a^b f_1(\tau_1)\mathrm{d}\tau_1 \dots f_n(\tau_n)\mathrm{d}\tau_n. \tag{2.7}$$

Proposition 2.2. *For all $f_1, \dots, f_n \in W$, $I(f_1, \dots, f_n; \tau)$ is well-defined, i.e.* (2.6) *is finite, and we have*

$$\frac{\partial}{\partial \tau}\Big|_{\tau=\tau_0} I(f_1, \dots, f_n; \tau) = -f_1(\tau_0) I(f_2, \dots, f_n; \tau_0). \tag{2.8}$$

Proof: [3], Lemma 4.5 and Proposition 4.7 i). □

The second part of the preceding proposition is the analogue for regularized iterated integrals of the differential equation satisfied by ordinary iterated integrals ([8], p.40). It will be crucial in the proof of linear independence of iterated Eisenstein integrals.

2.3. **Iterated integrals on the q-disk**. We have seen that $I(f_1, \dots, f_n; \tau)$ is a holomorphic function on the upper half-plane. Using the change of coordinates (2.4), we can rewrite $I(f_1, \dots, f_n; \tau)$ as a regularized iterated integral on the punctured q-disk

$$I(f_1, \dots, f_n; \tau) = \frac{1}{(2\pi i)^n} \sum_{i=0}^{n} \int_q^0 R[f_1|\dots|f_i]_{\frac{\mathrm{d}q}{q}} \int_q^1 [(f^\infty)_{i+1}|\dots|(f^\infty)_n]_{\frac{\mathrm{d}q}{q}}. \tag{2.9}$$

The virtue of representation (2.9) is that one sees that

$$I(f_1, \dots, f_n; \tau) \in W[\log(q)], \quad \log(q) := 2\pi i \tau, \tag{2.10}$$

and therefore every linear identity between the $I(f_1, \dots, f_n; \tau)$ reduces, by comparing coefficients, to a linear system of equations. Also, note that if all $f_i \in W_\mathbb{Q} := \mathbb{Q}[[q]] \cap W$, then $(2\pi i)^n I(f_1, \dots, f_n; \tau) \in W_\mathbb{Q}[\log(q)]$.

Definition 2.3. For $k_1, \dots, k_n \geq 0$, we define the *(indefinite, Hecke-normalized) iterated Eisenstein integral* to be

$$\mathcal{G}(2k_1, \dots, 2k_n; q) = (2\pi i)^n I(G_{2k_1}, \dots, G_{2k_n}; \tau) \in W_\mathbb{Q}[\log(q)]. \tag{2.11}$$

Note that by Proposition 2.2,

$$\begin{aligned} \frac{1}{2\pi i}\frac{\partial}{\partial \tau}\Big|_{\tau=\tau_0} \mathcal{G}(2k_1, \dots, 2k_n; q) &= q\frac{\partial}{\partial q}\Big|_{q=q_0} \mathcal{G}(2k_1, \dots, 2k_n; q) \\ &= -G_{2k_1}(q_0)\mathcal{G}(2k_2, \dots, 2k_n; q_0). \end{aligned} \tag{2.12}$$

Example 2.4. In length one, we have (cf. [3], Example 4.10)

$$\mathcal{G}(2k;q) = \frac{B_{2k}}{4k}\log(q) - \sum_{n\geq 1}\frac{\sigma_{2k-1}(n)}{n}q^n. \tag{2.13}$$

Later on, we will also need the integral over the non-constant part G_{2k}^0 of the Eisenstein series G_{2k}. We denote this by

$$\mathcal{G}^0(2k;q) := \int_q^0 G_{2k}^0(q_1)\frac{\mathrm{d}q_1}{q_1} = -\sum_{n\geq 1}\frac{\sigma_{2k-1}(n)}{n}q^n. \tag{2.14}$$

3 Proof of linear independence

Having defined iterated Eisenstein integrals in the last section, we now turn to the proof of their linear independence. The larger part of this section is devoted to proving linear independence over $\mathrm{Frac}(\mathbb{Z}[[q]])$, the fraction field of the ring of formal power series with integer coefficients. In order to achieve this, we use the following general linear independence result for iterated integrals, which is (a special case of) the main result of [5] (Theorem 2.1). Let X be an alphabet (not necessarily finite), and denote by X^* the free monoid on X.

Theorem 3.1 (Deneufchâtel, Duchamp, Minh, Solomon)**.** *Let* $(\mathcal{A},\mathrm{d})$ *be a differential algebra over a field* k *of characteristic zero, whose ring of constants* $\ker(\mathrm{d})$ *is precisely equal to* k*. Let* $\mathcal{C}$ *be a differential subfield of* $\mathcal{A}$ *(i.e. a subfield such that* $\mathrm{d}\mathcal{C}\subset\mathcal{C}$*). Suppose that* $S\in\mathcal{A}\langle\langle X\rangle\rangle$ *is a solution to the differential equation*

$$\mathrm{d}S = M\cdot S, \tag{3.1}$$

where $M=\sum_{x\in X}u_x x\in\mathcal{C}\langle\langle X\rangle\rangle$ *is a homogeneous series of degree* 1*, with initial condition* $S_1=1$*, where* S_1 *denotes the coefficient of the empty word in the series* S*. The following are equivalent:*

(i) *The family of coefficients* $(S_w)_{w\in X^*}$ *of* S *is linearly independent over* $\mathcal{C}$*.*

(ii) *The family* $\{u_x\}_{x\in X}$ *is linearly independent over* k*, and we have*

$$\mathrm{d}\mathcal{C}\cap\mathrm{Span}_k(\{u_x\}_{x\in X}) = \{0\}. \tag{3.2}$$

We are now in a position to prove our main result.

Theorem 3.2. *The family of iterated Eisenstein integrals* (2.11) *is linearly independent over* $\mathrm{Frac}(\mathbb{Z}[[q]])$*.*

Proof: We will apply Theorem 3.1 with the following parameters:

- $k=\mathbb{Q}$, $\mathcal{A}=\mathbb{Q}((q))[\log(q)]$ with differential $\mathrm{d}=q\frac{\partial}{\partial q}$, and $\mathcal{C}=\mathrm{Frac}(\mathbb{Z}[[q]])$ (the latter is a differential field by the quotient rule for derivatives)
- $X=\{a_{2k}\}_{k\geq 0}$, $u_{a_{2k}}=-G_{2k}(q)$, hence

$$M(\tau) = -\sum_{k\geq 0}G_{2k}(q)a_{2k}.$$

With these conventions, it follows from (2.12) that the formal series

$$1+\int_\tau^{i\infty}[M(\tau_1)]_{\mathrm{d}\tau}+\int_\tau^{i\infty}[M(\tau_1)|M(\tau_2)]_{\mathrm{d}\tau}+\ldots\in\mathcal{O}(\mathbb{H})\langle\langle X\rangle\rangle, \tag{3.3}$$

where the iterated integrals are regularized as in Section 2.2, is a solution to the differential equation $\mathrm{d}S = M \cdot S$, with $S_1 = 1$. Consequently, the coefficient of the word $w = a_{2k_1} \dots a_{2k_n}$ in S is equal to $\mathcal{G}(2k_1, \dots, 2k_n; q)$. Moreover, since $\mathbb{Q}$-linear independence of the Eisenstein series is well-known (cf. e.g. [16], VII.3.2), it remains to verify (3.2) in our situation.

To this end, assume that there exist $\alpha_{2k} \in \mathbb{Q}$, all but finitely many of which are equal to zero, such that

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) \in \mathrm{d}\mathcal{C}. \tag{3.4}$$

Clearing denominators, we may assume that $\alpha_{2k} \in \mathbb{Z}$. Furthermore, from the definition of $\mathrm{d} = q\frac{\partial}{\partial q}$, one sees that the image $\mathrm{d}\mathcal{C}$ of the differential operator d does not contain any constant except for zero. Therefore, the constant term $\sum_{k \geq 0} \alpha_{2k} G_{2k}^{\infty}$ of (3.4) vanishes; in other words

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) = \sum_{k \geq 1} \alpha_{2k} G_{2k}^{0}(q) \in q\mathbb{Q}[[q]]. \tag{3.5}$$

Now the differential d is invertible on $q\mathbb{Q}[[q]]$, and inverting d is the same as integrating. Hence (3.4) is equivalent to

$$\sum_{k \geq 1} \alpha_{2k} \mathcal{G}^{0}(2k; q) \in \mathcal{C}. \tag{3.6}$$

But this is absurd, unless all the α_{2k} vanish, as we shall see now. Indeed, if $f \in \mathcal{C} = \mathrm{Frac}(\mathbb{Z}[[q]])$, then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $f \in \mathbb{Z}[m^{-1}]((q))$. This follows from the well-known inversion formula for power series. On the other hand, the coefficient of q^p in $\mathcal{G}^0(2k; q)$, for p a prime number, is given by

$$\frac{\sigma_{2k-1}(p)}{p} = \frac{p^{2k-1}+1}{p} \equiv \frac{1}{p} \mod \mathbb{Z} \tag{3.7}$$

(cf. (2.14)). Thus, we must have

$$\frac{1}{p} \sum_{k \geq 1} \alpha_{2k} \in \mathbb{Z}[m^{-1}], \tag{3.8}$$

for every prime number p. But then the integer $\sum_{k \geq 1} \alpha_{2k}$ is divisible by infinitely many primes (namely, at least all the primes which don't divide m), which implies $\sum_{k \geq 1} \alpha_{2k} = 0$.

Now assume that k_1 is the smallest positive, even integer with the property that $\alpha_{k_1} \neq 0$. Consider the coefficient of $q^{p^{k_1}}$ in $\mathcal{G}^0(2k; q)$, which is equal to

$$\frac{\sigma_{2k-1}(p^{k_1})}{p^{k_1}} = \frac{1}{p^{k_1}} \sum_{j=0}^{k_1} p^{j(2k-1)} \equiv \begin{cases} \frac{1}{p^{k_1}} \mod \mathbb{Z} & \text{if } 2k > k_1 \\ \frac{1}{p^{k_1}} + \frac{1}{p} \mod \mathbb{Z} & \text{if } 2k = k_1 \end{cases} \tag{3.9}$$

(cf. (2.14)). By (3.6), $\frac{\alpha_{k_1}}{p} + \frac{1}{p^{k_1}} \sum_{k \geq 1} \alpha_{2k} \in \mathbb{Z}[m^{-1}]$, and by what we have seen before, $\sum_{k \geq 1} \alpha_{2k} = 0$. Hence

$$\frac{\alpha_{k_1}}{p} \in \mathbb{Z}[m^{-1}], \tag{3.10}$$

for every prime number p, which again implies $\alpha_{k_1} = 0$, in contradiction to our assumption $\alpha_{k_1} \neq 0$. Therefore, in (3.6), we must have $\alpha_{2k} = 0$ for all $k \geq 1$ and (3.2) is verified. □

Having established linear independence of iterated Eisenstein integrals over the field $\mathrm{Frac}(\mathbb{Z}[[q]])$, linear independence over $\mathbb{C}$ follows almost immediately.

Corollary 3.3. *The family of iterated Eisenstein integrals $\mathcal{G}(2k_1, \ldots, 2k_n; q)$ for $n \geq 0$ and all $k_i \geq 0$ is linearly independent over the complex numbers.*

Proof: Let $\mathcal{G}_1, \ldots, \mathcal{G}_n$ be iterated Eisenstein integrals, and assume we have a relation

$$\sum_{i=1}^{n} \alpha_i \mathcal{G}_i = 0 \tag{3.11}$$

with $\alpha_i \in \mathbb{C}$. Since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[[q]])$, it follows from Theorem 3.2 that the matrix of the coefficients of the $\mathcal{G}_i$, considered as series in $\log(q)^k q^l$ for $k, l \geq 0$, has maximal rank n. Therefore $\alpha_i = 0$ for $i = 1, \ldots, k$. $\square$

Remark 3.4. By the shuffle product formula, the $\mathbb{C}$-vector space spanned by the iterated Eisenstein integrals is a $\mathbb{C}$-algebra. Corollary 3.3 implies that it is a free shuffle algebra, and thus a polynomial algebra by [15].

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